

Algebraic structure of the Feynman propagator and a new correspondence for canonical transformations

Akihiro Ogura and Motoo Sekiguchi[†]

Laboratory of Physics, Nihon University, Matsudo 271-8587, Japan

and

[†]Faculty of Engineering, Kokushikan University, Tokyo 154-8515, Japan

Abstract

We investigate the algebraic structure of the Feynman propagator with a general time-dependent quadratic Hamiltonian system. Using the Lie-algebraic technique we obtain a normal-ordered form of the time-evolution operator, and then the propagator is easily derived by a simple “Integration Within Ordered Product” (IWOP) technique. It is found that this propagator contains a classical generating function which demonstrates a new correspondence between classical and quantum mechanics.

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1 Introduction

In recent years there has been considerable interest in the application of the Lie-algebraic technique in order to derive the propagators or density matrices of the systems. These systems include not only a free particle system but also a harmonic oscillator [1], a charged oscillator in a constant magnetic field [2], and a general time-dependent oscillator [3]. In those papers cited, the direct use of the position and momentum operators plays an important role. However, creation and annihilation operators are often used for representation of the Lie algebra and applied to harmonic oscillator systems. It is recognized [4] that the use of creation and annihilation operators clarifies the physics and simplifies the calculation.

The purpose of this paper is to present a method in which the Lie algebra of the squeezed operator plays a direct role in deriving the propagator of a general time-dependent quadratic Hamiltonian system. It is found that this propagator contains a classical generating function which implies a new correspondence between classical and quantum mechanics. In carrying out this program, we focus on a time-evolution operator with a general time-dependent quadratic Hamiltonian system as follows:

$$U(t) = \exp \left[-\frac{i}{2} \{ \alpha(t)p^2 + \beta(t)(qp + pq) + \gamma(t)q^2 \} \right], \quad (1)$$

where q and p are the coordinate and momentum variables and $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ are arbitrary real functions of time t . In the next section we first review squeezed operators and then calculate the propagator $\langle Q|U(t)|q \rangle$ with the aid of the “Integration Within Ordered Product” (IWOP) technique. In this case the creation and annihilation operators play a central role in deriving the propagator. In section 3, we investigate the character of the propagator from the point of linear transformations in the coordinate-momentum phase space. We then show a new correspondence between classical and quantum mechanics. Section 4 is devoted to a discussion.

2 Squeezed operator and the propagator

The squeezed operators [5, 6] are defined using the annihilation operator $a = (q + ip)/\sqrt{2}$ as follows:

$$K_+ = \frac{a^\dagger a^\dagger}{2}, \quad K_0 = \frac{a^\dagger a}{2} + \frac{1}{4}, \quad K_- = \frac{aa}{2}, \quad (2)$$

which form the $SU(1, 1)$ Lie algebra

$$[K_+, K_-] = -2K_0, \quad [K_0, K_\pm] = \pm K_\pm. \quad (3)$$

We rewrite the unitary operator (1) in terms of the squeezed operators (2)

$$U(t) = \exp [\tau(t)K_+ + i\sigma(t)K_0 - \tau^*(t)K_-], \quad (4)$$

where

$$\begin{cases} \tau(t) &= \beta(t) + i\frac{\alpha(t) - \gamma(t)}{2} \\ \sigma(t) &= -\alpha(t) - \gamma(t), \end{cases} \quad (5)$$

$\tau(t)$ is a complex function and $\sigma(t)$ is a real function of time t . Using the technique of differential equations [7, 8], we obtain the normal ordered form of the operator (4)

$$U(t) = \exp \left[-\frac{r(t)}{s(t)} K_+ \right] \exp [-2K_0 \ln s(t)] \exp \left[\frac{r^*(t)}{s(t)} K_- \right], \quad (6)$$

where $s(t)$ and $r(t)$ are defined by

$$\begin{cases} s(t) &= \cosh \Delta - \frac{i\sigma(t)}{2\Delta} \sinh \Delta \\ r(t) &= -\frac{\tau(t)}{\Delta} \sinh \Delta \end{cases} \quad (7)$$

with

$$\Delta^2 = |\tau(t)|^2 - \frac{\sigma^2(t)}{4} = \beta^2(t) - \alpha(t)\gamma(t) \quad (8)$$

which satisfy the following relation;

$$|s(t)|^2 - |r(t)|^2 = \cosh^2 \Delta - \sinh^2 \Delta = 1. \quad (9)$$

Now, we are in a position to calculate the Feynman propagator $\langle Q|U(t)|q \rangle$. The coherent state $|z \rangle$ is defined [9] by the eigenstate of the annihilation operator a with the complex eigenvalue z , i.e.

$$a|z \rangle = z|z \rangle, \quad (10)$$

and form the completeness relation

$$\int \frac{d^2 z}{2\pi i} |z \rangle \langle z| = 1, \quad (11)$$

where $\int d^2 z \equiv \int d[\text{Re}(z)]d[\text{Im}(z)]$. To obtain the Feynman propagator $\langle Q|U(t)|q \rangle$, we use the completeness relation (11) with arguments of z_1 and z_2

$$\langle Q|U(t)|q \rangle = \int \frac{d^2 z_1 d^2 z_2}{(2\pi i)^2} \langle Q|z_1 \rangle \langle z_1|U(t)|z_2 \rangle \langle z_2|q \rangle. \quad (12)$$

With the aid of the IWOP technique [10, 11],

$$\langle z_1|U(t)|z_2 \rangle = \frac{1}{\sqrt{s}} \exp \left[-\frac{r}{2s}(z_1^*)^2 + \frac{z_2 z_1^*}{s} + \frac{r^*}{2s}z_2^2 - \frac{|z_1|^2}{2} - \frac{|z_2|^2}{2} \right] \quad (13)$$

and the coherent state with coordinate representation

$$\langle z_2|q \rangle = \frac{1}{\pi^{1/4}} \exp \left[-\frac{q^2}{2} + \sqrt{2}qz_2^* - \frac{(z_2^*)^2}{2} - \frac{|z_2|^2}{2} \right], \quad (14)$$

$$\langle z_1|Q \rangle = \frac{1}{\pi^{1/4}} \exp \left[-\frac{Q^2}{2} + \sqrt{2}Qz_1^* - \frac{(z_1^*)^2}{2} - \frac{|z_1|^2}{2} \right], \quad (15)$$

we integrate z_1 and z_2 of (12) to obtain

$$\begin{aligned} \langle Q|U(t)|q \rangle &= \sqrt{\frac{1}{\pi(s-s^*-r+r^*)}} \times \\ &\exp \left[\frac{2qQ}{s-s^*-r+r^*} - \frac{q^2}{2} \frac{s+s^*-r-r^*}{s-s^*-r+r^*} - \frac{Q^2}{2} \frac{s+s^*+r+r^*}{s-s^*-r+r^*} \right]. \end{aligned} \quad (16)$$

This is the transition amplitude $q \rightarrow Q$ derived from the unitary operator (1).

3 Propagator and linear canonical transformations

In this section, we shall investigate the character of the propagator (16). Before doing so, we shall concentrate on the meaning of the operator (1) in terms of the coordinate-momentum phase space. Transforming the annihilation and creation operators under (6), we obtain the following form for the operator:

$$\begin{cases} U^\dagger(t)aU(t) &= s^*(t)a - r(t)a^\dagger \\ U^\dagger(t)a^\dagger U(t) &= s(t)a^\dagger - r^*(t)a, \end{cases} \quad (17)$$

which is described in the coordinate-momentum phase space

$$\begin{cases} Q(t) &= U^\dagger(t)qU(t) = A(t)q + B(t)p \\ P(t) &= U^\dagger(t)pU(t) = C(t)q + D(t)p, \end{cases} \quad (18)$$

where (Q, P) and (q, p) are the new and old quantum canonical variables which combine linearly with the time-dependent real functions $A(t)$, $B(t)$, $C(t)$, and $D(t)$. Also, these coefficients satisfy

$$\begin{cases} s(t) &= \frac{D(t)+A(t)}{2} + i\frac{B(t)-C(t)}{2} \\ r(t) &= \frac{D(t)-A(t)}{2} - i\frac{B(t)+C(t)}{2}, \end{cases} \quad (19)$$

subject to

$$|s(t)|^2 - |r(t)|^2 = A(t)D(t) - B(t)C(t) = 1, \quad (20)$$

which signifies the existense of the inverse of the linear transformation (18). On the other hand, by straightforward calculation $U^\dagger(t)qU(t)$ and $U^\dagger(t)pU(t)$ with the operator (1) using the Baker-Campbell-Hausdorff formulae, we can assign the coefficients $A(t)$, $B(t)$, $C(t)$, and $D(t)$ in terms of $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ as follows:

$$\begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix} = \begin{pmatrix} \cosh \Delta + \frac{\beta(t)}{\Delta} \sinh \Delta & \frac{\alpha(t)}{\Delta} \sinh \Delta \\ -\frac{\gamma(t)}{\Delta} \sinh \Delta & \cosh \Delta - \frac{\beta(t)}{\Delta} \sinh \Delta \end{pmatrix} \quad (21)$$

$$= \begin{pmatrix} \frac{s+s^*-r-r^*}{2} & \frac{-is+is^*+ir-ir^*}{2} \\ \frac{is-is^*+ir-ir^*}{2} & \frac{s+s^*+r+r^*}{2} \end{pmatrix}. \quad (22)$$

Note that these coefficients (21) and (22) are consistent with (7) and (19).

Now we rewrite the propagator (16) in terms of $A(t)$, $B(t)$, $C(t)$, and $D(t)$ from (22), then

$$\langle Q|U(t)|q \rangle = \sqrt{\frac{1}{2\pi i B(t)}} \exp[-iW(q, Q, t)], \quad (23)$$

where

$$W(q, Q, t) = \frac{qQ}{B(t)} - \frac{A(t)}{2B(t)}q^2 - \frac{D(t)}{2B(t)}Q^2. \quad (24)$$

This propagator is for the most general time-dependent quadratic Hamiltonian system, and thus results for any spacial case can easily be deduced from it. For example, we take the harmonic oscillator. In this case, we have

$$\alpha(t) = \frac{t}{m}, \quad \beta(t) = 0, \quad \gamma(t) = m\omega^2 t, \quad (25)$$

then $\Delta = i\omega t$ and so

$$\begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix} = \begin{pmatrix} \cos \omega t & \frac{1}{m\omega} \sin \omega t \\ -m\omega \sin \omega t & \cos \omega t \end{pmatrix} \quad (26)$$

We recover the well known result for the Feynman propagator

$$\langle Q|U(t)|q \rangle = \sqrt{\frac{m\omega}{2\pi i \sin \omega t}} \exp \left[i \left\{ -\frac{m\omega}{\sin \omega t} qQ + \frac{m\omega}{2 \tan \omega t} (q^2 + Q^2) \right\} \right] \quad (27)$$

In the limit $\omega \rightarrow 0$, we fall back to the free particle case; i.e. $\alpha(t) = \frac{t}{m}$, $\beta(t) = \gamma(t) = 0$. Then, we obtain

$$\begin{pmatrix} A(t) & B(t) \\ C(t) & D(t) \end{pmatrix} = \begin{pmatrix} 1 & \frac{t}{m} \\ 0 & 1 \end{pmatrix} \quad (28)$$

and

$$\langle Q|U(t)|q \rangle = \sqrt{\frac{m}{2\pi i t}} \exp \left[i \left\{ -\frac{m}{t} q Q + \frac{m}{2t} (q^2 + Q^2) \right\} \right]. \quad (29)$$

It is worth mentioning two points here. The first point is that the exponential function $W(q, Q, t)$ in (24) is a generating function that gives rise to a classical linear canonical transformation (18) with an ordinary prescription in classical mechanics;

$$p = \frac{\partial W}{\partial q}, \quad P = -\frac{\partial W}{\partial Q}. \quad (30)$$

It was Dirac who first discussed that the exponential of the classical generating function can be used as the quantum transformation function [12, 13, 14]. This remark lead Feynman to his path integral formulation of quantum mechanics [15]. The second point is that we have derived the propagator (23) with a general time-dependent quadratic Hamiltonian system (1), which reflects a linear transformation in coordinate-momentum phase space (18). Let us consider the reverse, once we have the linear transformation (18) in coordinate-momentum phase space, the propagator is written down at once in the form (23). These two points reveal a new correspondence between classical and quantum mechanics.

4 Summary

We have presented a method by which the propagator of a general time-dependent quadratic Hamiltonian system can be derived by the Lie-algebraic technique for the squeezed operators. It was found that this propagator contains a generating function which gives rise to a linear transformation in coordinate-momentum phase space in classical mechanics. Furthermore, our formulation has a useful attribute in that once we have the linear canonical transformation in coordinate-momentum phase space, its quantum counterpart of a unitary operator and the Feynman propagator are easily calculated.

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